TOWARDS THE THEORY OF BENNEY EQUATIONS

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Abstract

The solutions of the Benney equations are constructed. Their properties are discussed.

1 Introduction

A famous Benney equations describing a long waves on the surface of fluid has the form

$$\frac{\partial f(x,v,t)}{\partial t}+v\frac{\partial f(x,v,t)}{\partial x}-\frac{\partial A_0(x,t)}{\partial x}\frac{\partial f(x,v,t)}{\partial v}=0, \tag{1}$$

where $A_0(x,t)=\int_{-\infty}^{+\infty} f(x,v,t)dv$

It can be rewritten in the form of nonlinear system of p.d.e.

$$\frac{\partial f(x,v,t)}{\partial t}+v\frac{\partial f(x,v,t)}{\partial x}-\left(\frac{\partial g(x,v,t)}{\partial x}+\frac{\partial h(x,v,t)}{\partial x}\right)\frac{\partial f(x,v,t)}{\partial v}=0,$$

$$\frac{\partial g(x,v,t)}{\partial v}=f(x,v,t),\quad \frac{\partial h(x,v,t)}{\partial v}=-f(x,v,t), \tag{2}$$

where

$$g(x,v,t)=\int_{-\infty}^{v} f(x,v,t)dv,\quad h(x,v,t)=\int_{v}^{\infty} f(x,v,t)dv.$$

From the system (2) we get the relations

$$\frac{\partial}{\partial x}g(x,v,t)+\frac{\partial}{\partial x}h(x,v,t)=\frac{\frac{\partial}{\partial t}f(x,v,t)+v\frac{\partial}{\partial x}f(x,v,t)}{\frac{\partial}{\partial v}f(x,v,t)}$$

and

$$\frac{\partial}{\partial v}g(x,v,t)+\frac{\partial}{\partial v}h(x,v,t)=0,$$

which is equivalent the nonlinear p.d.e.

$$\left(\frac{\partial}{\partial v}f(x,v,t)\right)\frac{\partial^2}{\partial t\partial v}f(x,v,t)+\left(\frac{\partial}{\partial v}f(x,v,t)\right)\frac{\partial}{\partial x}f(x,v,t)+\left(\frac{\partial}{\partial v}f(x,v,t)\right)v\frac{\partial^2}{\partial v\partial x}f(x,v,t)-$$

$$-\left(\frac{\partial^2}{\partial v^2}f(x,v,t)\right)\frac{\partial}{\partial t}f(x,v,t)-\left(\frac{\partial^2}{\partial v^2}f(x,v,t)\right)v\frac{\partial}{\partial x}f(x,v,t)=0 \tag{3}$$

is followed.

To construction of particular solutions of this equation can be used the method of the (u,v)-transformation developed first by author .

2 The method of solution

To integrate the partial nonlinear differential equation

$$F(x, y, z, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xz}, f_{yz}, f_{zz}) = 0 \qquad (4)$$

can be applied a following method.

We use the change of the functions and variables according to the rule

$$f(x, y, z) \to u(x, t, z), \quad y \to v(x, t, z), \quad f_x \to u_x - \frac{v_x}{v_t} u_t, \quad f_z \to u_z - \frac{v_z}{v_t} u_t, \quad f_y \to \frac{u_t}{v_t}, ... \qquad (5)$$

In result instead of the equation (4) one get the relation between the new variables $u(x, t, z)$ and $v(x, t, z)$ and their partial derivatives

$$\Phi(u, v, u_x, u_t, u_z, v_x, v_t, v_z, ...) = 0. \qquad (6)$$

In some cases the integration of the last equation is more simple problem than integration of the equation (4).

To illustrate this method let us consider some of examples.

1.

The equation

$$\frac{\partial}{\partial x} z(x, y) - \left(\frac{\partial}{\partial y} z(x, y)\right)^2 = 0 \qquad (7)$$

is transformed into the following form

$$\frac{\partial}{\partial x} u(x, t) - \frac{\left(\frac{\partial}{\partial t} u(x, t)\right) \frac{\partial}{\partial x} v(x, t)}{\frac{\partial}{\partial t} v(x, t)} - \frac{\left(\frac{\partial}{\partial t} u(x, t)\right)^2}{\left(\frac{\partial}{\partial t} v(x, t)\right)^2} = 0.$$

Using the substitution

$$u(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad v(x, t) = \frac{\partial}{\partial t} \omega(x, t)$$

we find the equation for $\omega(x, t)$

$$\frac{\partial}{\partial x} \omega(x, t) + t^2 = 0.$$

Its integration lead to

$$\omega(x, t) = -t^2 x + F_1(t)$$

where $F_1(t)$ is arbitrary function.

Now with help of $\omega(x, t)$ we find the functions $u(x, t)$ and $v(x, t)$

$$u(x, t) = -t^2 x + t \frac{d}{dt} F_1(t) - F_1(t), \quad v(x, t) = -2\, tx + \frac{d}{dt} F_1(t)$$

or

$$u(x, t) = ty + t^2 x - F_1(t), \quad y = -2\, tx + \frac{d}{dt} F_1(t).$$

After the choice of arbitrary function $F_1(t)$ and elimination of the parameter t from these relations we get the function $z(x, y)$, satisfying the equation (7).

2.

The equation

$$\frac{\partial}{\partial x} z(x, y) + z(x, y) \left(\frac{\partial}{\partial y} z(x, y)\right) = 0 \qquad (8)$$

is transformed into the following form

$$\frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\frac{\partial}{\partial x}v(x,t)}{\frac{\partial}{\partial t}v(x,t)} + u(x,t)\frac{\left(\frac{\partial}{\partial t}u(x,t)\right)}{\left(\frac{\partial}{\partial t}v(x,t)\right)} = 0.$$

Using the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

we find the equation for $\omega(x,t)$

$$\left(\frac{\partial}{\partial t}\omega(x,t)\right)t^2 - \frac{\partial}{\partial x}\omega(x,t) - t\omega(x,t) = 0.$$

Its integration give us the function

$$\omega(x,t) = tF\left(\frac{1-tx}{t}\right)$$

where $F()$ is arbitrary function.

Now with help of the function $\omega(x,t)$ we can find the functions $u(x,t)$ and $v(x,t)$. Then after the choice of arbitrary function $F()$ and elimination of the parameter t from the relations

$$z = u(x,t), \quad y = v(x,t)$$

we can get the function $z(x,y)$, satisfying the equation (8).

By analogy the substitution

$$\left\{u(x,t) = \frac{\partial}{\partial t}\omega(x,t), \quad v(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t)\right\}$$

into the equation for $\omega(x,t)$ lead to the equation

$$\frac{\partial}{\partial x}\phi(x,t) + \frac{\partial}{\partial t}\phi(x,t) = 0$$

with general solution

$$\phi(x,t) = F(t-x)$$

which also give us the solution of the equation (8).

3.

The equation meeting in theory of the Benney equations

$$\frac{\partial^2}{\partial y\partial z}f(x,y,z) + \left(\frac{\partial}{\partial y}f(x,y,z)\right)\frac{\partial^2}{\partial x\partial y}f(x,y,z) - \left(\frac{\partial}{\partial x}f(x,y,z)\right)\frac{\partial^2}{\partial y^2}f(x,y,z) = 0 \tag{9}$$

after the (u,v)-transformation with the change $(y \to t)$ takes the form

$$\left(\frac{\partial^2}{\partial t\partial z}u(x,t,z)\right)\left(\frac{\partial}{\partial t}v(x,t,z)\right)^2 - \left(\frac{\partial^2}{\partial t^2}u(x,t,z)\right)\left(\frac{\partial}{\partial z}v(x,t,z)\right)\frac{\partial}{\partial t}v(x,t,z) +$$

$$+\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial}{\partial z}v(x,t,z)\right)\frac{\partial^2}{\partial t^2}v(x,t,z) - \left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial^2}{\partial t\partial z}v(x,t,z)\right)\frac{\partial}{\partial t}v(x,t,z) +$$

$$+\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial^2}{\partial t\partial x}u(x,t,z)\right)\frac{\partial}{\partial t}v(x,t,z) - \left(\frac{\partial}{\partial t}u(x,t,z)\right)^2\frac{\partial^2}{\partial t\partial x}v(x,t,z) -$$

$$-\left(\frac{\partial}{\partial x}u(x,t,z)\right)\left(\frac{\partial}{\partial t}v(x,t,z)\right)\frac{\partial^2}{\partial t^2}u(x,t,z)+\left(\frac{\partial}{\partial x}u(x,t,z)\right)\left(\frac{\partial}{\partial t}u(x,t,z)\right)\frac{\partial^2}{\partial t^2}v(x,t,z)=0.$$

In result of substitution of the form

$$u(x,t,z)=t\frac{\partial}{\partial t}\omega(x,t,z)-\omega(x,t,z),\quad v(x,t,z)=\frac{\partial}{\partial t}\omega(x,t,z)$$

we get from this relation the linear p.d.e. with the respect of the function $\omega(x,t,z)$

$$\frac{\partial^2}{\partial t\partial z}\omega(x,t,z)+t\frac{\partial^2}{\partial t\partial x}\omega(x,t,z)-\frac{\partial}{\partial x}\omega(x,t,z)=0$$

Its solutions can be obtained by the Laplace method and after elimination of the parameter t from the expressions for the function $(f(x,y,z)\to u(x,t,z))$ and $(y\to v(x,t,z))$ the solutions of initial equation (9) can be constructed.

By analogy this method can be applied to obtaining particular solutions of the equation (3).

3 The (u,v)-transformation of the equation (3)

For convenience we write the equation (3) in the form

$$\left(\frac{\partial}{\partial y}f(x,y,z)\right)\frac{\partial^2}{\partial y\partial z}f(x,y,z)+\left(\frac{\partial}{\partial y}f(x,y,z)\right)\frac{\partial}{\partial x}f(x,y,z)+\left(\frac{\partial}{\partial y}f(x,y,z)\right)y\frac{\partial^2}{\partial x\partial y}f(x,y,z)-$$
$$-\left(\frac{\partial^2}{\partial y^2}f(x,y,z)\right)\frac{\partial}{\partial z}f(x,y,z)-\left(\frac{\partial^2}{\partial y^2}f(x,y,z)\right)y\frac{\partial}{\partial x}f(x,y,z)=0, \tag{10}$$

where instead of the variables v and t we used the variables y and z.

After the (u,v)-transformation with the change of the variable y on parameter t the equation (10) takes the form of the relation between the functions $u(x,t,z)$ and $v(x,t,z)$ and their derivatives

$$\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial^2}{\partial t\partial z}u(x,t,z)\right)\frac{\partial}{\partial t}v(x,t,z)-\left(\frac{\partial}{\partial t}u(x,t,z)\right)^2\frac{\partial^2}{\partial t\partial z}v(x,t,z)+$$
$$+\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial}{\partial t}v(x,t,z)\right)^2\frac{\partial}{\partial x}u(x,t,z)-\left(\frac{\partial}{\partial t}u(x,t,z)\right)^2\left(\frac{\partial}{\partial t}v(x,t,z)\right)\frac{\partial}{\partial x}v(x,t,z)+$$
$$+\left(\frac{\partial}{\partial t}u(x,t,z)\right)v(x,t,z)\left(\frac{\partial^2}{\partial t\partial x}u(x,t,z)\right)\frac{\partial}{\partial t}v(x,t,z)-\left(\frac{\partial}{\partial t}u(x,t,z)\right)^2v(x,t,z)\frac{\partial^2}{\partial t\partial x}v(x,t,z)-$$
$$-\left(\frac{\partial^2}{\partial t^2}u(x,t,z)\right)\left(\frac{\partial}{\partial t}v(x,t,z)\right)\frac{\partial}{\partial z}u(x,t,z)+\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial^2}{\partial t^2}v(x,t,z)\right)\frac{\partial}{\partial z}u(x,t,z)-$$
$$-v(x,t,z)\left(\frac{\partial^2}{\partial t^2}u(x,t,z)\right)\left(\frac{\partial}{\partial t}v(x,t,z)\right)\frac{\partial}{\partial x}u(x,t,z)+v(x,t,z)\left(\frac{\partial}{\partial t}u(x,t,z)\right)\left(\frac{\partial^2}{\partial t^2}v(x,t,z)\right)\frac{\partial}{\partial x}u(x,t,z)=0.$$

There are a lot possibilities to bring this relation to one equation. Classification all types of reductions is open problem.

We use a simplest type of reductions.

As example after the substitution

$$u(x,t,z)=t\frac{\partial}{\partial t}\omega(x,t,z)-\omega(x,t,z),\quad v(x,t,z)=\frac{\partial}{\partial t}\omega(x,t,z). \tag{11}$$

this relation lead to the nonlinear partial differential equation

$$-t\left(\frac{\partial}{\partial x}\omega(x,t,z)\right)\frac{\partial^2}{\partial t^2}\omega(x,t,z)+\left(\frac{\partial}{\partial t}\omega(x,t,z)\right)\frac{\partial}{\partial x}\omega(x,t,z)-t\frac{\partial^2}{\partial t\partial z}\omega(x,t,z)+\frac{\partial}{\partial z}\omega(x,t,z)-$$

$$-\left(\frac{\partial}{\partial t}\omega(x,t,z)\right)t\frac{\partial^2}{\partial t\partial x}\omega(x,t,z)=0. \tag{12}$$

Its particular solutions can be used for construction of solutions of the equation (10).

Let as consider some examples.

Using the substitution

$$\omega(x,t,z)=A(t,z)+Bxt \tag{13}$$

we get from the (12) the equation

$$-t^2B\frac{\partial^2}{\partial t^2}A(t,z)-t\frac{\partial^2}{\partial t\partial z}A(t,z)+\frac{\partial}{\partial z}A(t,z)=0$$

with general solution

$$A(t,z)=\_F1(\frac{zB-\ln(t)}{B})+\_F2(z)t \tag{14}$$

dependent from two arbitrary functions $\_F2(z)$ and $\_F1(\frac{zB-\ln(t)}{B})$.

In particular case

$$\_F1(-\frac{-zB+\ln(t)}{B})=\frac{(-zB+\ln(t))^2}{B^2}$$

with the help of the formulaes (11, 13) we find the relations

$$f(x,y,z)B^2+2\,zB-2\,\ln(t)+z^2B^2-2\,zB\ln(t)+(\ln(t))^2=0,$$

$$ytB^2+2\,zB-2\,\ln(t)-\_F2(z)tB^2-B^3xt=0.$$

From last equation we get the expression for parameter t

$$t=e^{-LambertW(1/2\,B^2(-y+\_F2(z)+Bx)e^{zB})+zB}$$

and after substitution its into the first one we find the function $f(x,y,z)$

$$f(x,y,z)=-\frac{1+LambertW(1/2\,B^2\,(-y+\_F2(z)+Bx)\,e^{zB})}{B}$$

which is solution of the equation (10).

Substitution

$$v(x,t,z)=t\frac{\partial}{\partial t}\omega(x,t,z)-\omega(x,t,z),\quad u(x,t,z)=\frac{\partial}{\partial t}\omega(x,t,z)$$

lead to the equation

$$t\left(\frac{\partial}{\partial x}\omega(x,t,z)\right)\frac{\partial^2}{\partial t^2}\omega(x,t,z)+\frac{\partial^2}{\partial t\partial z}\omega(x,t,z)+\left(\frac{\partial}{\partial t}\omega(x,t,z)\right)t\frac{\partial^2}{\partial t\partial x}\omega(x,t,z)-$$

$$-\left(\frac{\partial^2}{\partial t\partial x}\omega(x,t,z)\right)\omega(x,t,z)=0. \tag{15}$$

It has the particular solution defined by the expression

$$\omega(x,t,z)=P(t,z)+xe^{-z} \tag{16}$$

where the function $P(t,z)$ satisfies the equation

$$te^{-z}\frac{\partial^2}{\partial t^2}P(t,z)+\frac{\partial^2}{\partial t\partial z}P(t,z)=0,$$

having general solution dependent from two arbitrary functions

$$P(t,z) = \_F2(z) + \int \_F1(-\ln(t) - e^{-z})dt. \tag{17}$$

In particular case

$$\_F1(-\ln(t) - e^{-z}) = -\ln(t) - e^{-z}$$

we find the solution of the equation (10)

$$f(x,y,z) = \frac{-\ln(-ye^z - \_F2(z)e^z - x)e^z + ze^z - 1}{e^z}.$$

Remark 1 *The equation (15) after the substitution*

$$\omega(x,t,z) = A(t, x-z) = A(t,\eta)$$

takes the form

$$t\left(\frac{\partial}{\partial\eta}A(t,\eta)\right)\frac{\partial^2}{\partial t^2}A(t,\eta) + t\left(\frac{\partial}{\partial t}A(t,\eta)\right)\frac{\partial^2}{\partial\eta\partial t}A(t,\eta) - \left(\frac{\partial^2}{\partial\eta\partial t}A(t,\eta)\right)A(t,\eta) - \frac{\partial^2}{\partial\eta\partial t}A(t,\eta) = 0$$

and can be solved exactly.

In fact, the function $B(t,\eta) = A(t,\eta) + 1$ satisfies the equation

$$t\left(\frac{\partial}{\partial\eta}B(t,\eta)\right)\frac{\partial^2}{\partial t^2}B(t,\eta) + t\left(\frac{\partial}{\partial t}B(t,\eta)\right)\frac{\partial^2}{\partial\eta\partial t}B(t,\eta) - \left(\frac{\partial^2}{\partial\eta\partial t}B(t,\eta)\right)B(t,\eta) = 0$$

which admits the first integral

$$\left(B(t,\eta) - t\frac{\partial}{\partial t}B(t,\eta)\right)\frac{\partial}{\partial\eta}B(t,\eta) - K(\eta) = 0, \tag{18}$$

where $K(\eta)$ is arbitrary.

Equation (18) is in form

$$\left(\frac{\partial}{\partial y}h(x,y)\right)h(x,y) - x\left(\frac{\partial}{\partial x}h(x,y)\right)\frac{\partial}{\partial y}h(x,y) - K(y) = 0. \tag{19}$$

After the (u,v) - transformation it is reduced to the relation

$$\left(\frac{\partial}{\partial t}u(x,t)\right)u(x,t)\frac{\partial}{\partial t}v(x,t) - x\left(\frac{\partial}{\partial t}u(x,t)\right)\left(\frac{\partial}{\partial x}u(x,t)\right)\frac{\partial}{\partial t}v(x,t) + x\left(\frac{\partial}{\partial t}u(x,t)\right)^2\frac{\partial}{\partial x}v(x,t) -$$

$$-K(v(x,t))\left(\frac{\partial}{\partial t}v(x,t)\right)^2 = 0$$

which is equivalent the first order nonlinear p.d.e. with respect to the function $\omega(x,t)$ at the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

$$-t^2\frac{\partial}{\partial t}\omega(x,t) + t\omega(x,t) - xt\frac{\partial}{\partial x}\omega(x,t) + K\left(\frac{\partial}{\partial t}\omega(x,t)\right) = 0. \tag{20}$$

Solutions of the equation (20) depend from the function $K(\omega_t)$.

As example in the case

$$K(\omega_t) = \frac{\partial \omega(x,t)}{\partial t}$$

we get the linear equation

$$-t^2 \frac{\partial}{\partial t}\omega(x,t) + t\omega(x,t) - xt\frac{\partial}{\partial x}\omega(x,t) + \frac{\partial}{\partial t}\omega(x,t) = 0$$

with general solution

$$\omega(x,t) = \_F1\left(\frac{t^2-1}{x^2}\right)x.$$

Remark 2 *Legendre-transformation of the equation (19)*

$$h(x,y) = \xi\frac{\partial}{\partial \xi}\theta(\xi,\rho) + \rho\frac{\partial}{\partial \rho}\theta(\xi,\rho) - \theta(\xi,\rho), \quad \frac{\partial}{\partial x}h(x,y) = \xi, \quad \frac{\partial}{\partial y}h(x,y) = \rho,$$

$$x = \frac{\partial}{\partial \xi}\theta(\xi,\rho), \quad y = \frac{\partial}{\partial \rho}\theta(\xi,\rho)$$

lead to the equation

$$\rho^2\frac{\partial}{\partial \rho}\theta(\xi,\rho) - \rho\,\theta(\xi,\rho) - K(\frac{\partial}{\partial \rho}\theta(\xi,\rho)) = 0$$

which is reduced to the Bernoulli equation after application of the suitable Legendre transformation and so it is integrable.

With the help of solutions of the equation (19) the functions $A(t,\eta)$ and $\omega(x,t,z) = A(t,x-z) = A(t,\eta)$ can be determined.

Then after elimination of the parameter t from the expressions for the functions $\omega(x,t,z)$ and $y(x,t,z)$ particular solutions of the equation (10) can be constructed.

4 On the equations of shallow water waves

The equations of shallow water waves

$$\frac{\partial}{\partial y}h(x,y) + g(x,y)\frac{\partial}{\partial x}h(x,y) + h(x,y)\frac{\partial}{\partial x}g(x,y) = 0,$$

$$\frac{\partial}{\partial y}g(x,y) + g(x,y)\frac{\partial}{\partial x}g(x,y) + \frac{\partial}{\partial x}h(x,y) = 0 \tag{21}$$

are intimately connected with the Benney equation.

We apply the method of (u,v)-transformation for construction some particular solutions of the system (21).

For this purpose we use the substitution

$$h(x,y) = \frac{\partial}{\partial x}f(x,y), \quad g(x,y) = -\frac{\frac{\partial}{\partial y}f(x,y)}{\frac{\partial}{\partial x}f(x,y)}$$

and replace the system (21) by the equation with respect the function $f(x,y)$

$$-\left(\frac{\partial^2}{\partial y^2}f(x,y)\right)\left(\frac{\partial}{\partial x}f(x,y)\right)^2 + 2\left(\frac{\partial}{\partial y}f(x,y)\right)\left(\frac{\partial^2}{\partial x\partial y}f(x,y)\right)\frac{\partial}{\partial x}f(x,y) -$$

$$-\left(\frac{\partial}{\partial y}f(x,y)\right)^2\frac{\partial^2}{\partial x^2}f(x,y) + \left(\frac{\partial^2}{\partial x^2}f(x,y)\right)\left(\frac{\partial}{\partial x}f(x,y)\right)^3 = 0. \tag{22}$$

The u, v)-transformation of the equation (22) with the conditions

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t) \tag{23}$$

lead to the equation with respect the function $\omega(x,t)$

$$-\left(\frac{\partial^2}{\partial t^2}\omega(x,t)\right)\left(\frac{\partial^2}{\partial x^2}\omega(x,t)\right)\left(\frac{\partial}{\partial x}\omega(x,t)\right)^3 - \left(\frac{\partial^2}{\partial t^2}\omega(x,t)\right)t^2\frac{\partial^2}{\partial x^2}\omega(x,t) -$$
$$-2\,t\left(\frac{\partial^2}{\partial t\partial x}\omega(x,t)\right)\frac{\partial}{\partial x}\omega(x,t) + \left(\frac{\partial}{\partial x}\omega(x,t)\right)^2 + t^2\left(\frac{\partial^2}{\partial t\partial x}\omega(x,t)\right)^2 +$$
$$+\left(\frac{\partial^2}{\partial t\partial x}\omega(x,t)\right)^2\left(\frac{\partial}{\partial x}\omega(x,t)\right)^3 = 0. \tag{24}$$

Particular solution of the equation (24) depending from one arbitrary function is

$$\omega(x,t) = A(t) + xB(t),$$

where the function $A(t)$ is arbitrary, and the function $B(t)$ satisfies the equation

$$-2\,t\left(\frac{d}{dt}B(t)\right)B(t) + (B(t))^2 + t^2\left(\frac{d}{dt}B(t)\right)^2 + \left(\frac{d}{dt}B(t)\right)^2(B(t))^3 = 0$$

having the solution of the form

$$B(t) = -\left(1/6\sqrt[3]{54\,t + \_C1^3 + 6\sqrt{81\,t^2 + 3\,t\_C1^3}} + 1/6\,\frac{\_C1^2}{\sqrt[3]{54\,t + \_C1^3 + 6\sqrt{81\,t^2 + 3\,t\_C1^3}}} + 1/6\,\_C1\right)^2. \tag{25}$$

On the basis of this solution and in deciding the arbitrary function $A(t)$ we define the function $\omega(x,t)$ and then after elimination of the parameter t from the relations (23) the solutions of the equation (22) can be constructed.

As example in particular case from (25) is followed

$$\_C1 = 0, \quad B(t) = -1/2\,\sqrt[3]{2}t^{2/3}$$

and choosing the function $A(t)$ in the form

$$A(t) = \sqrt[3]{t}$$

we get the relations

$$6\,f(x,y)t^{2/3} + 4\,t - t^{4/3}\sqrt[3]{2}x = 0,$$
$$3\,yt^{2/3} - 1 + \sqrt[3]{2}x\sqrt[3]{t} = 0.$$

Elimination of the parameter t from these relations lead to the equation for the function $f(x,y)$

$$-3\,2^{2/3}x^2 - 108\,\sqrt[3]{2}xyf(x,y) + 324\,y^2\,(f(x,y))^2 - 48\,y - 12\,f(x,y)x^3 = 0 \tag{26}$$

which is the simplest particular solution of the equation (22).

In deciding on the function $A(t)$ in more general form can be obtained more complicated solutions of the equation (22) and the corresponding system (21).

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